

Neither Name, Nor Number

FEDERICO HOLIK

December 21, 2011

Instituto de Astronomía y Física del Espacio (IAFE)
Casilla de Correo 67, Sucursal 28, 1428 Buenos Aires, Argentina

Abstract

Since its origins, Quantum mechanics has presented problems with the concept of individuality. It is argued that quantum particles do not have individuality, and so, one can speak about “entities without identity”. On the contrary, we claim that the problem of quantum non individuality goes deeper, and that one of its most important features is the fact that there are quantum systems for which particle number is not well defined. In this work, we continue this discussion in relation to the problem about the one and the many.

Key words: quasisets, particle number, quasicardinality, quantum indistinguishability.

1 Introduction

The concept of individuality in quantum mechanics clashes radically with its classical counterpart. In classical physics, particles can be considered as individuals without giving rise to consistency problems but, in quantum mechanics this is not the case. Problems arise if one intends to individuate elementary particles. The responses to this problem range from the claim that there are no elementary particles at all to the assertion that there are particles but they are intrinsically indistinguishable (i.e., indistinguishable in an ontological sense). Some authors talk about “entities without identity” [6]. In this work, we claim that the problem of quantum non individuality is even worse: quantum non individuality clashes with the concept of number. In this work we discuss the significance of the superpositions of states with different particle number in the Fock-space formalism. We continue to interpret them as systems with an ontologically undefined particle number. And we conclude that quantum systems not only suffer the loss of transcendental identity; they also lose the property of having a definite number. A quantum system is not a *one*, nor a *many*. But in spite of it, it *is*.

In relation to the problem of quantum indistinguishability, Michael Readhead and Paul Teller claim in [14] that:

“Interpreters of quantum mechanics largely agree that classical concepts do not apply without alteration or restriction to quantum objects. In Bohr’s formulation this means that one cannot simultaneously apply complementary concepts, such as position and momentum, without restriction. In particular, this means that one cannot attribute classical, well defined trajectories to quantum systems. But in a more fundamental respect it would seem that physicists, including Bohr, continue to think of quantum objects classically as individual things, capable, at least conceptually, of bearing labels. It is this presumption and its implications which we need to understand and critically examine.” M. Readhead and P. Teller ([14], p.202)

When individuality of quanta is studied exhaustively, most investigations seem to point in the direction that quanta have no individuality at all (see for example [6] for a detailed analysis). To put it in Schrödinger’s words:

“I mean this: that the elementary particle is not an individual; it cannot be identified, it lacks ‘sameness’. The fact is known to every physicist, but is rarely given any prominence in surveys readable by nonspecialists. In technical language it is covered by saying that the particles ‘obey’ a newfangled statistics, either Einstein-Bose or Fermi-Dirac statistics. [...] The implication, far from obvious, is that the unsuspected epithet ‘this’ is not quite properly applicable to, say, an electron, except with caution, in a restricted sense, and sometimes not at all.” E. Schrödinger ([17], p.197)

This work should be considered as a tentative to enlarge the problem posed in the quotation of Readhead and Teller cited above. We claim that it is usually supposed that quantum systems can be considered as singular unities, (a quantum system as a “one”), or collections of them (a quantum system as a “many”). In this work we want to “understand and critically examine” this assumption, thus continuing the developments in [5].

In section 2, we discuss the meaning of superpositions of particle number eigenstates in Fock-space. In 3 we discuss the assumptions and limitations of the notion of particles aggregate. Then, in 4, we review the approach of quasiset theory to quantum indistinguishability. In section 5, we review the modifications to Quasiset-theory made in [3] and in 6 we discuss an enlargement of the Manin’s problem.

2 Quantity in Quantum Mechanics

As is well known, performing a single measurement in a quantum system does not allow to attribute the result of this measurement to a property which the system possesses before the measurement is performed without giving rise to serious problems [10]. What is the relationship between this fact and the quantity of particles in a quantum system? Take for example the electromagnetic field (with a single frequency for simplicity) in the following state:

$$|\psi\rangle = \alpha |1\rangle + \beta |2\rangle \quad (1)$$

where $|1\rangle$ and $|2\rangle$ are eigenvectors of the particle number operator with eigenvalues 1 and 2 respectively, and α and β are complex numbers which satisfy $|\alpha|^2 + |\beta|^2 = 1$. If a measurement of the number of particles of the system is performed, one or two particles will be detected, with probabilities $|\alpha|^2$ and $|\beta|^2$ respectively. Any other possibility is excluded. Suppose that in a single measurement two particles are detected. What allows us to conclude that the system had two particles before the measurement was performed? The assertion that the number of particles is varying in time because particles are being constantly created and destroyed is also problematic, because it assumes that at each instant the number of particles is well defined. Only in case that it is known with certainty that the system is in an eigenstate of the particle number operator we can say that the system has a well defined cardinal. There would be no problem too if it is known with certainty that the system is prepared in an statistical mixture. In this case, the corresponding density operator would be:

$$\rho_m = |\alpha|^2 |1\rangle\langle 1| + |\beta|^2 |2\rangle\langle 2| \quad (2)$$

where the subindex “m” stands for statistical mixture. But the density operator corresponding to (1) is:

$$\rho = (\alpha |1\rangle + \beta |2\rangle)(\alpha^* \langle 1| + \beta^* \langle 2|) \quad (3)$$

which is the same as:

$$\rho = |\alpha|^2 |1\rangle\langle 1| + |\beta|^2 |2\rangle\langle 2| + \alpha\beta^* |1\rangle\langle 2| + \alpha^*\beta |2\rangle\langle 1| \quad (4)$$

The presence of interference terms in the last equation implies that difficulties will appear in stating that, after a single measurement, the system has the quantity of particles obtained as the result of the measurement. In this case, the incapability of knowing the particle number would not come from our ignorance about the system, but from the fact that in this state, the particle number is not even well defined. We see thus, that if superpositions of particle number are allowed, we are faced with an *indeterminancy* in the particle number. An indeterminacy of the same kind of that which appears in position or spin. But this time, the “property” affected is very special; *it is just the property linked to the dichotomy of being one or many.*

3 How many?

Let us discuss the origin of the concept of “particle number”. We start posing the questions: In which sense do we talk about quantum systems composed, for example, of a single photon? How do we decide if the field is in a single photon state or not? What do we mean when we use the words “single photon”? We could search for a clue to answer these questions in our laboratory experience, i.e., making measurements. In experiments, we often use a picture which allows us to speak about the photon as a particle (and so, as an individual). In a similar way, and always in relation to experiments, we talk about the other particles (electrons, protons, etc.). But in a deeper analysis we find that this supposed “particle behaviour” of quantum systems

is hardly compatible with its classical counterpart, and though experiments seem to suggest an idea of individuality, it is well established that this does not enable us to consider particles as individuals. Individuality is not compatible with the formal structure of quantum mechanics [5] and so elementary particles cannot be considered as individuals, as E. Schrödinger pointed out in the early days of the theory [17]. In spite of these difficulties, we continue speaking about photons, electrons, etc., using a jargon which has a lot of points in common with classical physics, source of conceptual confusion.

But it is just this interpretation, which presupposes the concept of ‘particle’ (or quantum object, to put it in more general terms) which lies at the heart of the notion of ‘particles aggregate’ (and so, a *defined* particle number, or a *defined* number of objects). There are definite experimental arrangements which force the appearance of particle characteristics as a final result of a single process. Experiments are designed to find out which is the particle number, but as we have already mentioned in 2, this does not imply that the resulting number is a property that pertains to the system under study. On the contrary, it refers to the definite *process* which takes place in each measurement. We are not allowed to consider the system as an aggregate of individuals as if they were classical objects, simply because the notion of “object” is incompatible with the formal structure of the theory [5]. The fall of the notion of “object”, causes the fall of the notion of “objects aggregate”.

We claim that superpositions in particle number discussed in 2 are a direct expression of the fall of the concept of “objects aggregate”. We know that it is possible to assign to some quantum systems an associated number, take for example the electrons of a Litium atom, or single photon states. But particle number superpositions show that in general, it is not true that a definite number can be always assigned in a consistent manner. We are accustomed in our classical experience to assign always a definite number to the set of things that we are studying. But when we are faced to a quantum system, we are forced to abandon that habit.

4 Quasi-set theory and indistinguishability

In the standard approach to quantum indistinguishability, particles are labeled as if they were individuals, and then indistinguishability is recovered via symmetrization postulates [14]. This is a variant of what in [7] was called the Weyl’s strategy. Many authors pointed out the importance of developing alternative ways to describe quantum indistinguishability, reproducing the results obtained by standard techniques, but assuming in every step of the deduction that elementary particles of the same class are intrinsically indistinguishable from the beginning (see, for example, [6], [7] and [12]), without making appeal to Weyl’s strategy variants. Another claim is that quantum mechanics does not possess its own language, but it uses a portion of functional analysis which is itself based on set theory, and thus finally related to classical experience. This statement was posed by Y. Manin [9], the Russian mathematician who suggested that standard set theories (as Zermelo-Fraenkel, ZF) are influenced by every day experience, and so it would be interesting to search for set theories which inspire its concepts in the quantum domain. This is known as Manin’s problem [8]. In this spirit, and looking for a solution to Manin’s problem a quasiset theory (\mathcal{Q} in the following) was developed [7], [16].

Quasiset theory seems to be adequate to represent as “sets” of some kind (quaisets) the collections of truly indistinguishable entities. This aim is reached in \mathcal{Q} because equality is not a primitive concept, and there exist certain kinds of *urelemente* (m-atoms) for which only an indistinguishability relationship applies. So, in \mathcal{Q} , non individuality is incorporated by proposing the existence of entities for which it has no sense to assert that they are identical to themselves or different from others of the same class.

\mathcal{Q} contains a copy of Zermelo-Fraenkel set theory plus Urelemente (ZFU). These Urele-

mente are called M-atoms. This feature divides the theory in two parts. One region involves only the elements of ZFU , and the other one contains quasisets whose elements can be truly indistinguishable entities. Quasisets containing only indistinguishable elements are called “pure quasisets”. The ZFU copy of quasisets is called “the classical part of the theory ” in [16]. Indistinguishability is modeled in this theory using a primitive binary relation \equiv (indistinguishability) and a new class of atoms, called m-atoms, which express the existence of quanta in the theory [16]. So, in the frame of \mathcal{Q} , when we speak of m-atoms of the same class, the only thing that we can assert about them is that they are indistinguishable, and nothing else makes sense, for expressions like $x = y$ are not well formed formulas. This is to say that we cannot make assertions about their identity, i.e., it has no sense to say that an m-atom is equal or different of other m-atom of the same class. It is important to remark that in \mathcal{Q} , indistinguishability does not imply identity, and so it is possible that even being indistinguishable, two m-atoms belong to different quasisets, thus avoiding the collapse of indistinguishability in classical identity [16].

\mathcal{Q} is constructed in such a way that allows the existence of collections of truly indistinguishable objects, and thereof it is impossible to label the elements of pure quasisets. For this reason, the construction used to assign cardinals to sets of standard ZFU theories cannot be applied any more. But even if electrons are indistinguishable (in an ontological sense), every physicist knows that it makes sense to assert that, for example, a Litium atom has three electrons. It is for that reason that \mathcal{Q} should allow quasisets to have some kind of associated cardinal. In \mathcal{Q} this is solved postulating that a cardinal number is assigned to every quasiset (remember that there is a copy of ZFU in \mathcal{Q}). Some other properties of the standard cardinal are postulated too. This rule for the assignment of cardinals uses a unary symbol $qc()$ as a primitive concept. So in \mathcal{Q} , the quasicardinal is a primitive concept alike ZF , in which the property that to every set corresponds a single cardinal number can be derived from the axioms [11].

An important theorem of \mathcal{Q} is related to the unobservability of permutations:

[Unobservability of Permutations] Let x be a finite quasi-set such that x does not contain all indistinguishable from z , where z is an m -atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists w' such that

$$(x - z') \cup w' \equiv x$$

It is the assertion in the language of \mathcal{Q} that permutation of indistinguishable quanta cannot imply any observable effect, or put in words of Penrose:

“[a]ccording to quantum mechanics, any two electrons must necessarily be completely identical [in the physicist’s jargon, that is, indistinguishable], and the same holds for any two protons and for any two particles whatever, of any particular kind. This is not merely to say that there is no way of telling the particles apart; the statement is considerably stronger than that. If an electron in a person’s brain were to be exchanged with an electron in a brick, then the state of the system would be *exactly the same state* as it was before, not merely indistinguishable from it! The same holds for protons and for any other kind of particle, and for the whole atoms, molecules, etc. If the entire material content of a person were to be exchanged with the corresponding particles in the bricks of his house then, in a strong sense, nothing would be happened whatsoever. What distinguishes the person from his house is the *pattern* of how his constituents are arranged, not the individuality of the constituents themselves” [15, p. 32].

In the next section, we go back to the problem of an undefined particle number and we relate it with \mathcal{Q} .

5 Modifications to the theory of Quasi-sets

The way in which the quasicardinal is introduced in \mathcal{Q} implies that every quaset has an associated cardinal, i.e., every quaset has a well defined number of elements. But the idea that an aggregate of entities must necessarily have an associated number which represents the number of entities is based in our every day experience. As we have mentioned in section 2, there are quantum systems to which it is not allowed to assign a number of particles in a consistent manner.

Taking into account these considerations, it is worth asking: is it possible to represent a system prepared in the state (4) in the frame of quaset theory? Which place would correspond to a system like (4) in that theory? If such system could be represented as a quaset, then it should have an associated quasicardinal, for every quaset has it. But this does not seem to be proper, considering what we have discussed in section 2. It follows that it does not appear reasonable to assign a quasicardinal to every quaset if quaset theory has to include all bosonic and fermionic systems (in all their possible many particle states). Therefore, a system in the state (4) cannot be included in \mathcal{Q} as a quaset. Yet, it would be interesting to study the possibility of including systems in those states (such as (4)) in the formalism.

A possible way out would be the introduction of a Fock-space, but constructed using the non classical part of \mathcal{Q} . This option has been considered in [4]. In that paper, we reformulate Fock-space quantum mechanics using \mathcal{Q} . Due to the theorem of unobservability of permutations (see last section of this work), we avoid the use of the *labeled-tensor-product-Hilbert-space formalism* (LTPHSF), as called by Redhead and Teller [13], [14]. So we can express states such as (4) as superpositions of different particle number in this novel space.

Another possibility is to reformulate \mathcal{Q} in such a way that the quasicardinal is not to be taken as a primitive concept, but as a derived one, turning it into a property that some quasets have and some others do not (in analogy with the property “being a prime number” of the integers). Those quasets for which the property of having a quasicardinal is not satisfied, would be suitable to represent quantum systems with particle number not defined (such as equation (4)). This property would also fit well with the position that asserts that particle interpretation is not adequate in, for example, quantum electrodynamics. With such a modification of \mathcal{Q} , for example, a field (in any state) could always be represented as a quaset, avoiding the necessity of regarding the field as a collection of classical “things”. On the contrary, the field would be described by a quaset which has a defined quasicardinal only in special cases, but not in general. Thus a field would be represented as something which genuinely is nor a one nor a many. And for that reason this quaset could not be interpreted so simply as a collection of particles (because it seems reasonable to assume that a collection of particles, indistinguishable or not, must always have a well defined particle number).

We have followed the idea of modifying \mathcal{Q} in [3]. In that work, we have searched for a \mathcal{Q} -like theory in which the quasicardinal is not taken as a primitive concept alike \mathcal{Q} . We showed that it is possible to develop a theory about collections of indistinguishable entities in which quasicardinal is not a primitive concept. We define a singleton which allows us to extract “just one element” from a given quaset X , and obtain a subquaset X^- such that:

$$X \supset X^-$$

It is important to remark that it has no sense to ask which is the extracted element, because this query is not defined in a theory without identity. It could happen that $X^- = \emptyset$ or not. If $X^- \neq_E \emptyset$ it follows that we can make the same operation again, and obtain X^{--} . Then we have:

$$X \supset X^- \supset X^{--}$$

Going ahead with this process, it could be the case that this chain of inclusions stopped (in case the last quaset so obtained be the empty quaset), or that it has no end. So we could conceive two qualitatively different situations:

Situation 1:

$$X \supset X^- \supset X^{--} \supset X^{---} \supset \dots\dots$$

(the inclusions chain continues indefinitely)

Situation 2:

$$X \supset X^- \supset X^{--} \supset X^{---} \supset \dots\dots \supset \emptyset$$

(the inclusion chain ends in the empty quaset).

We call *descendent chains* to the chains of inclusions which appear in situations 1 and 2. In order to grant that one of these situations holds for every quaset we postulate the existence of descendent chains. This is done by first translating the concept of descendent chain to first order language:

Definition 5.1

$$CD_X(\gamma) \longleftrightarrow$$

$$(\gamma \in \wp(\wp(X)) \wedge X \in \gamma \wedge \forall z \forall y (z \in \gamma \wedge y \in \gamma \wedge z \neq_E y \longrightarrow (z \supset y \vee y \supset z)) \\ \wedge \forall z (z \in \gamma \wedge z \neq_E \emptyset \longrightarrow \exists y (y \in \gamma \wedge DD_z(y) \wedge \forall w (w \in \gamma \wedge DD_z(w) \longrightarrow w =_E y))))$$

$CD_X(\gamma)$ is read as: γ is a descendant chain of X .

In the construction shown in [3] we have reobtained that every (finite) quaset has a well defined quasicardinal. But in that construction, we can assert that a quaset has a definite quasicardinal only if it satisfies the definition of finiteness, and nothing can be said about the quasets which do not satisfy the definition. Though not necesarilly useful for the problem of the undefined particle number, the axiomatic variant exposed in [3] shows explicitly that in a theory about collections of indistinguishable entities, the quasicardinal needs not be necessarily taken as a primitive concept. This result encourages the research of more complex axiomatic formulations, able to incorporate the quantum systems with undefined particle number as sets of some kind, thus enriching Manin's problem.

6 A new turn on Manin's problem

In this article, we have disscussed that quantum systems not only seem to lack individuality, but moreover, they seem to be undefinite in their cardinality. What is the meaning of this assertion? Perhaps, we have to abandon the supposition that we are dealing with something like "entities without identity", in the sense that we cannot consider a quantum system as a "one" or a "collection of ones". Perhaps, besides the ontological presupposition of individuality, we have to abandon the presupposition of number: so, we would have *neither name, nor number*. In this sense, the suggestions of Manin:

"We should consider the possibilities of developing a totally new language to speak about infinity.¹ Classical critics of Cantor (Brouwer *et al.*) argued that, say, the general choice axiom is an illicit extrapolation of the finite case.

¹Set theory is also known as the theory of the 'infinite'.

I would like to point out that this is rather an extrapolation of common-place physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behavior. Even ‘sets’ of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the ‘set’ of grains of sand. In general, a highly probabilistic ‘physical infinity’ looks considerably more complicated and interesting than a plain infinity of ‘things’.

should be enlarged in order to include the question of undefined number. It would be very interesting to search for a “set theory” inspired on quantum phenomena, which could take into account the problem of undefined cardinality (perhaps, the term “set” would not be appropriate any more in a theory like this). It is quite possible that the development of a theory like this one, would yield a new way to approach the problem of quantum separability. Perhaps, when we speak of a quantum system composed of two subsystems we are thinking of it *as a collection of things*, and this could be linked to the problems which emerge in quantum separability, because we have a wrong way to speak about it.

\mathcal{Q} gives an answer to Manin’s problem, in the sense that it is a “set theory” concerning collections of truly indistinguishable objects. But in order to solve the extended version of Manin’s problem, further formal developments needs to be achieved. The axiomatic variant presented in [3] shows that in a theory like \mathcal{Q} , the quantity of elements needs not to be a primitive concept. This result encourages the search for a theory in which it is impossible to assign a quasicardinal to certain quasisets in a consistent manner, thus allowing to describe what it seems to happen with some quantum systems, in which non individuality expresses itself in the fact that particle number is not defined, besides ontological indistinguishability.

Acknowledgements The author wish to thank Graciela Domenech and Christian de Ronde for useful discussions. This work was partially supported by the following grants: PICT 04-17687 (ANPCyT), PIP N° 6461/05 (CONICET), UBACyT N° X204.

References

- [1] Dalla Chiara, M. L. and Toraldo di Francia, G.: “Identity Questions from Quantum Theory”, in Gavroglu, K. *et. al.*, (eds.), *Physics, Philosophy and the Scientific Community*, Dordrecht, Kluwer Academic Publishers, pp. 39-46, (1995).
- [2] Dalla Chiara, M. L., Giuntini, R. and Krause, D., “Quasiset Theories for Microobjects: A Comparison”, in Castellani, E. (ed.), *Interpreting bodies: Classical and quantum objects in modern physics*, Princeton University Press, Princeton, pp. 142-152, (1998).
- [3] Domenech, G. and Holik, F., “A discussion on particle number and quantum indistinguishability”, *Foundations of Physics*, **37**, pp. 855-878, (2007).
- [4] Domenech, G., Holik, F. and Krause, D., “Q-spaces and the Foundations of Quantum Mechanics”, submitted to *Foundations of Physics*, (2008).
- [5] Domenech, G., Holik, F. and de Ronde, C., “Entities, Identity and the Formal Structure of Quantum Mechanics”, in press, (2008).
- [6] French, S. and Krause, D., *Identity in Physics: A historical, Philosophical, and Formal Analysis*, Oxford University Press (2006).

- [7] Krause, D., “Why quasi-sets?”, *Boletim da Sociedade Paranaense de Matematica* **20**, 73-92 (2003).
- [8] Manin, Yu. I., “Mathematical Problems I: Foundations”, in Browder, F. E. (ed.) (1976):, p. 36, cited in [6].
- [9] Manin, Yu. I., “A course in mathematical logic”, Springer-Verlag, p. 84 (1977), cited in [6].
- [10] Mittelstaedt, P., *The interpretation of quantum mechanics and the measurement process*, Cambridge University Press, Cambridge, (1998).
- [11] Halmos P., *Naive Set Theory*, D. Van Nostrand Company (1963).
- [12] Post, H., “Individuality and physics”, *The listener* **70**, 534-537; reprinted in *Vedanta for East and West* **32**, pp. 14-22, (1963), cited in [6].
- [13] Redhead, M. and Teller, P., “Particles, particle labels, and quanta: the toll of unacknowledged metaphysics”, *Foundations of Physics* **21**, pp. 43-62, (1991).
- [14] Redhead, M. and Teller, P., “Particle labels and the theory of indistinguishable particles in quantum mechanics”, *British Journal for the Philosophy of Science* **43**, pp. 201-218. pp. 14-22, (1992).
- [15] Penrose, R.: *The emperors new mind*, Oxford, Oxford Un. Press, (1989).
- [16] Santorelli, A., Krause, D. and Sant’Anna, A., “A critical study on the concept of identity in Zermelo-Fraenkel like axioms and its relationship with quantum statistics”, *Logique & Analyse* **189-192**, pp. 231-260, (2005).
- [17] Schrödinger, E., “What is an elementary particle?”, reprinted in Castellani, E. (ed.), *Interpreting bodies: classical and quantum objects in modern physics*, Princeton Un. Press, pp. 197-210, (1998).